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COHOMOLOGICAL CHARACTERISATIONS OF FINITE SOLVABLE AND NILPOTENT GROUPS

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1. Introduction

In this paper we give cohomological characterisations of finite solvable, supersolvable and nilpotent groups. These characterisation theorems involve the cohomology of G with simple kG -modules as coefficients where k is a finite prime field.

Other cohomological characterisations of nilpotent groups can be found in the literature, see for example [9]. Also, the Theorem of Huppert–Thompson–Tate (see for example [14], p. 93) can be viewed as a homological characterisation of nilpotent groups. Generalizations of this latter theorem can be found in Quillen [12] and in Stammbach [15]. Cohomological characterisations of solvable and supersolvable groups do not seem to appear in the literature.¹

We consider only finite groups G . Throughout the paper we fix a prime p and denote by k the field of p elements. Whenever necessary we regard k as a trivial kG -module. In our theorems we obtain characterisations of p -solvable, p -supersolvable and p -nilpotent groups. If our hypotheses are satisfied for all primes p dividing the group order we clearly obtain characterisations of solvable, supersolvable and nilpotent groups. Our Theorem A deals with the case of p -solvable groups (see [10], p. 659).

Theorem A. *Let G be a finite group. For any kG -module M let $C(M)$ denote its centralizer in G . Then G is p -solvable if and only if $H^1(G/C(M), M) = 0$ for all simple kG -modules M .*

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¹ After completion of this paper the author has been informed that D.W. Barnes has independently obtained the equivalence (i) \Leftrightarrow (ii) of Theorem C.

We remark that the “only if” part is due to Gaschütz (unpublished). Our Theorem B gives more information about the p -chief factors of G (see [10], p. 685).

Theorem B. *Let G be a finite group and let $\mathfrak{M} = \{M_1, \dots, M_n\}$ be a complete set of simple kG -modules. Let $\mathfrak{M} = \mathfrak{M}' \cup \mathfrak{M}''$ be a partition of \mathfrak{M} satisfying the hypothesis*

(H) *if $M' \in \mathfrak{M}'$ and $M'' \in \mathfrak{M}''$, then $\text{Ext}_{kG}^1(M', M'') = 0 = \text{Ext}_{kG}^1(M'', M')$.*

Then the following statements are equivalent.

- (i) *G is p -solvable and all its p -chief factors belong to \mathfrak{M}' ;*
- (ii) *$H^1(G/C(M'), M') = 0$ for all $M' \in \mathfrak{M}'$,
 $H^1(G, M'') = 0$ for all $M'' \in \mathfrak{M}''$;*
- (iii) *$H^i(G/C(M'), M') = 0$ for all $M' \in \mathfrak{M}'$ and $i = 1, 2, \dots$; $H^i(G, M'') = 0$ for all $M'' \in \mathfrak{M}''$ and $i = 0, 1, \dots$.*

Our Theorem C deals with the case of p -supersolvable groups (see [10], p. 716).

Theorem C. *Let G be a finite group. Then the following statements are equivalent*

- (i) *G is p -supersolvable,*
- (ii) *$H^1(G, M) = 0$ for all simple kG -modules of k -dimension $d \geq 2$,*
- (iii) *$H^i(G, M) = 0$ for all simple kG -modules of k -dimension $d \geq 2$ and $i = 0, 1, \dots$.*

Finally our Theorem D deals with the case of p -nilpotent groups (see [10], p. 427).

Theorem D. *Let G be a finite group. Then the following statements are equivalent*

- (i) *G is p -nilpotent,*
- (ii) *$H^1(G, M) = 0$ for all non-trivial simple kG -modules M ,*
- (iii) *$H^i(G, M) = 0$ for all non-trivial simple kG -modules M and all $i \geq 0$.*

In our proofs we shall need a homological characterisation of the simple modules belonging to a block (see Corollary 1). From this and as an application of our theorems on p -supersolvable and p -nilpotent groups we then easily obtain some results on the block structure of those groups (Corollaries 2, 3, 4, 5).

2. Solvable groups

Our Theorem A follows from Lemmas 1 and 2 below. Lemma 1 is due to Gaschütz (unpublished).

Lemma 1 (Gaschütz). *Let $G \neq e$ be p -solvable and let M be a faithful kG -module. Then $H^i(G, M) = 0$ for $i \geq 0$.*

Proof. Since M is faithful, the group G cannot contain any non-trivial normal p -subgroup (see [10], p. 485). Thus G contains a non-trivial normal p' -subgroup N . Consider the spectral sequence associated with the group extension $N \rightarrowtail G \twoheadrightarrow G/N$

$$H^s(G/N, H^t(N, M)) \Rightarrow H^{s+t}(G, M).$$

Since N and M are of coprime order we have $H^s(N, M) = 0$ for $s \geq 1$, so that

$$H^i(G, M) = H^i(G/N, H^0(N, M)) = H^i(G/N, M^N).$$

Now M^N is a kG -submodule of M . Since M is faithful and simple it follows that $M^N = 0$. This completes the proof of Lemma 1.

Lemma 2. If $H^1(G/C(M), M) = 0$ for all simple kG -modules M , then G is p -solvable.

Proof. We consider the group G of smallest order which satisfies the hypothesis of Lemma 2, but which is not p -solvable. We claim that G cannot be simple. It certainly cannot be cyclic of prime order, for any such group is p -solvable. If G is simple and non-abelian, then every non-trivial kG -module M is faithful, so that by hypothesis $H^1(G, M) = 0$. Since $H^1(G, k) = 0$ also, we conclude that $H^1(G, A) = 0$ for any finite dimensional kG -module A . But then it follows by dimension shifting that $H^i(G, A) = 0$ for all $i \geq 1$. This implies that p does not divide the group order (see for example [16]). Thus G would be a p' -group, hence p -solvable. This is a contradiction.

We consider then a minimal normal subgroup $e \neq N \neq G$. Then N cannot be p -solvable, for otherwise G/N would not be p -solvable, would satisfy the hypothesis of Lemma 2 and would be of smaller order than G . Since N is not p -solvable and is of smaller order than G there exists a simple kN -module B such that $H^1(N/C(B), B) \neq 0$.

In particular we have that $B \neq k$. The beginning of the five term sequence

$$0 \rightarrow H^1(N/C(B), B) \rightarrow H^1(N, B) \rightarrow \dots$$

associated with $C(B) \rightarrowtail N \twoheadrightarrow N/C(B)$ then shows that $H^1(N, B) \neq 0$.

We now consider $A = \text{Hom}_N(kG, B)$. We have

$$H^1(G, A) = H^1(G, \text{Hom}_N(kG, B)) = H^1(N, B) \neq 0.$$

By induction on the length of the G -composition series of A there is a composition factor A' of A such that $H^1(G, A') \neq 0$. Hence by hypothesis on G the centralizer D of A' is non-trivial. Of course the module A' considered as kN -module is completely reducible, the simple summands being kN -modules conjugate to B . The centralizers of these are certain G -conjugates of $C(B) \subset N$. Since B is a non-trivial module, $C(B) \neq N$. It follows that the centralizer D of A' intersects N in a G -normal subgroup S which is contained in some G -conjugate of $C(B)$. Since N is

minimal, S must be trivial. Thus G/D contains a copy of N and hence cannot be p -solvable. On the other hand it clearly satisfies the hypothesis of Lemma 2 and is of smaller order than G . This contradicts the minimality of G .

We remark that it is enough to consider simple kG -modules M lying in the principal kG -block. For suppose M does not lie in the principal kG -block, then clearly M does not lie in the principal $k(G/C(M))$ -block. Hence automatically $H^i(G, M) = H^i(G/C(M), M) = 0$ for all $i \geq 0$ (see for example [2], p. 178).

3. The chief factors of a solvable group

The following is a generalisation due to Cossey and Gaschütz [3] of a result of Fong and Gaschütz [5] (see also [6]). For completeness we include a cohomological proof of it.

Proposition 1. *Let G be a group and let M be an abelian chief factor of G with $p \nmid |M|$. Then M is in the principal block of kG .*

Proof. Let $N \supseteq \bar{N}$ be two normal subgroups of G with $N/\bar{N} \cong M$ and let

$$N/\bar{N} \rightarrow G/\bar{N} \twoheadrightarrow G/N \tag{3.1}$$

be the corresponding group extension. Consider the 5-term sequence associated with (3.1) for M as coefficient module

$$0 \rightarrow H^1(G/N, M) \rightarrow H^1(G/\bar{N}, M) \rightarrow \text{Hom}_G(M, M) \xrightarrow{\delta_E} H^2(G/N, M) \rightarrow \dots$$

Since $\text{Hom}_G(M, M) \neq 0$, we have that $H^1(G/\bar{N}, M) \neq 0$ or $H^2(G/N, M) \neq 0$. In either case M is in the principal p -block of some quotient of G . By the definition of blocks as two-sided ideals in the group algebra it immediately follows that M belongs to the principal p -block of G .

Remark. If M is complemented in G , i.e. if the extension (3.1) splits, then $\delta_E(1_M) = 0$ (see [8], p. 209). Hence $H^1(G/\bar{N}, M) \neq 0$, so that $H^1(G, M) \neq 0$. We have been unable to prove or disprove the statement that for any M in the principal block there is an $i \geq 1$ such that $H^i(G, M) \neq 0$.

Proposition 2. *Let $\mathfrak{M} = \mathfrak{M}' \cup \mathfrak{M}''$ be a partition of the simple kG -modules, satisfying*

$$(H) \quad \text{if } M' \in \mathfrak{M}' \text{ and } M'' \in \mathfrak{M}'', \text{ then } \text{Ext}_{kG}^1(M', M'') = 0 = \text{Ext}_{kG}^1(M'', M').$$

Then $M' \in \mathfrak{M}'$ and $M'' \in \mathfrak{M}''$ do not belong to the same block of kG .

Proof. Suppose they do, then there exists a sequence of simple modules $M' = M_1, M_2, \dots, M_n = M''$ such that in the sequence of the associated principal projec-

tive modules $P' = P_1, P_2, \dots, P_n = P''$ the modules P_i and P_{i+1} contain a common composition factor (see [4], p. 378). It will thus be enough to show that the composition factors of a principal projective P belong either all to \mathfrak{M}' or all to \mathfrak{M}'' . Suppose this is not true. Then let

$$P = Q_0 \supset Q_1 \supset \dots \supset Q_l = 0$$

be a composition series. Let $Q_0/Q_1 = \bar{M} \in \mathfrak{M}'$ and let $i \geq 1$ be the smallest integer such that $Q_i/Q_{i+1} = \bar{\bar{M}} \in \mathfrak{M}''$. Using (H) we see that $Q_0/Q_{i+1} \cong \bar{M} \oplus Q_0/Q_i$. But then P would have \bar{M} and $\bar{\bar{M}}$ as epimorphic images. This is a contradiction.

We say that two simple kG -modules M_1, M_2 are *related*, $M_1 \sim M_2$, if $\text{Ext}_{kG}^1(M_1, M_2) \neq 0$ or $\text{Ext}_{kG}^1(M_2, M_1) \neq 0$. We then consider the equivalence relation " \approx " generated by " \sim ". We have

Corollary 1. *Two simple kG -modules are in the same equivalence class under the equivalence relation \approx if and only if they belong to the same block.*

Proof. It is clear that if $\text{Ext}_{kG}^1(M_1, M_2) \neq 0$ then M_1, M_2 belong to the same block. This proves one direction. To prove the converse we suppose that M_1, M_2 belong to the same block but not to the same equivalence class. Consider the equivalence class \mathfrak{M}' of M_1 and define \mathfrak{M}'' to be the union of the other equivalence classes. By the definition of equivalence classes the partition $\mathfrak{M} = \mathfrak{M}' \cup \mathfrak{M}''$ satisfies the hypothesis (H). By Proposition 2 the modules in the same block as M_1 belong to \mathfrak{M}' . But $M_2 \in \mathfrak{M}''$. This is a contradiction.

We now continue by giving a *proof* of Theorem B.

We first prove (i) \Rightarrow (ii). It is clear from Theorem A that for $M \in \mathfrak{M}'$ we have $H^i(G/C(M), M) = 0$ for $i \geq 1$. By Propositions 1 and 2 the simple modules in the principal block of kG belong to \mathfrak{M}' . Hence the modules of \mathfrak{M}'' do not belong to the principal block by Proposition 2. Thus $H^i(G, M'') = 0$ for $M'' \in \mathfrak{M}''$.

The implication (iii) \Rightarrow (ii) is trivial.

In order to prove (ii) \Rightarrow (i) we first note that $H^1(G, M'') = 0$ implies $H^1(G/C(M''), M'') = 0$. This follows easily from the beginning of the 5-term sequence associated with $C(M'') \twoheadrightarrow G \twoheadrightarrow G/C(M'')$. We may thus conclude from Theorem A that G is p -solvable. Since the p -chief factors all belong to the principal block by Proposition 1 we see that they either belong all to \mathfrak{M}' or to \mathfrak{M}'' . We have to prove that they belong to \mathfrak{M}' . Suppose they belong to \mathfrak{M}'' , then the modules in \mathfrak{M}' would not belong to the principal block, so that $H^1(G, M') \neq 0$ for all $M' \in \mathfrak{M}'$. But then the homology of G with any kG -module as coefficients would be trivial, and it would follow that G is a p' -group (see [16]). Since a p' -group does not have non-trivial p -chief factors this completes our proof of Theorem B.

We remark that in our proof of Theorem B we have used the hypothesis (H) only to show that the simple modules in the principal block either all belong to \mathfrak{M}' or \mathfrak{M}'' .

Of course in the statement of Theorem B we can replace hypothesis (H) by this weaker one.

4. Supersolvable groups

Here we prove Theorem C.

In order to prove (ii) \Rightarrow (i) we invoke Theorem B. Let $\mathcal{M} = \mathcal{M}' \cup \mathcal{M}''$ be the partition of the set of isomorphism classes of simple kG -modules, where \mathcal{M}' consists of the simple kG -modules of dimension one and \mathcal{M}'' consists of the simple kG -modules of dimension $d \geq 2$. We shall prove that this partition satisfies (H). We first remark that for $M' \in \mathcal{M}'$ and $M'' \in \mathcal{M}''$ we have

$$\text{Ext}_{kG}^1(M', M'') = H^1(G, \text{Hom}(M', M'')).$$

Since M' is one dimensional and M'' is simple of dimension $d \geq 2$, $\text{Hom}(M', M'')$ is again simple of dimension d . Thus if (ii) is satisfied, our partition satisfies (H).

In order to be able to apply Theorem B it remains to prove that $H^1(G/C(M'), M') = 0$. But if M' is one dimensional, G acts via the multiplicative group k^* of k . Hence $|G/C(M')|$ is a divisor of $p - 1$. But in that case $|G/C(M')|$ and $|M'|$ are relatively prime, so that $H^1(G/C(M'), M') = 0$.

Since (iii) \Rightarrow (ii) is trivial, it remains to prove (i) \Rightarrow (iii). Thus let G be p -supersolvable and let M be a simple kG -module of dimension $d \geq 2$. Let $C = C(M)$. Then we have $H^i(G/C, M) = 0$ for $i \geq 1$ by Theorem A. Choose a G -composition series

$$C = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_t \supseteq N_{t+1} = 0.$$

We then proceed by induction on the group order. Thus we suppose that

$$H^i(G/N_i, M) = 0, \quad i \geq 1.$$

Consider then the extension $N_i \twoheadrightarrow G \twoheadrightarrow G/N_i$ and the associated spectral sequence

$$H^r(G/N_i, H^s(N_i, M)) \Rightarrow H^{r+s}(G, M).$$

If N_i is a p' -group, then $H^s(N_i, M) = 0$ for $s \geq 1$, so that

$$H^r(G, M) = H^r(G/N_i, M) = 0.$$

If $N_i \cong M'$ is a kG -module of dimension one, then

$$H^s(N_i, M) = \text{Hom}(H_s(M', k), M) = \bar{M}.$$

Moreover $H_s(M', k)$ is again of k -dimension one, so that $\bar{M} = \text{Hom}(H_s(M', k), M)$ is again simple of k -dimension $d \geq 2$. Hence $H^r(G/N_i, \bar{M}) = 0$. This implies that $H^{r+s}(G, M) = 0$. We note as a consequence of Theorem C and Corollary 1

Corollary 2. *A finite group G is p -supersolvable if and only if the principal block of kG contains only one dimensional simple modules.*

As a further application we prove

Corollary 3. *Let G be p -supersolvable and let k_1, \dots, k_r be the non-isomorphic simple modules in the first block. Then every block of G contains at most r non-isomorphic simple modules.*

Proof. We shall prove that if $\text{Ext}_{kG}^1(M_1, M_2) \neq 0$ then $M_2 \cong k_i \otimes M_1$ for some i . By our characterisation of blocks (Corollary 1) and using the fact that $k_i \otimes k_j \otimes M \cong k_l \otimes M$ for some l , this is clearly enough. Thus let $\text{Ext}_{kG}^1(M_1, M_2) = H^1(k, \text{Hom}(M_1, M_2)) \neq 0$. It follows that the module $A = \text{Hom}(M_1, M_2)$ contains a composition factor belonging to the first block. Let

$$A = A_0 \supset A_1 \supset \dots \supset A_n = 0$$

be a composition series and let l be the largest integer with $A_l/A_{l+1} \cong k_i$ for some i . Then $\text{Ext}_{kG}^1(k_i, A_{l+1}) = 0$, so that $A_l \cong k_i \oplus A_{l+1}$. It then easily follows that $\text{Hom}_{kG}(k_i, A) \neq 0$. But we have $\text{Hom}_{kG}(k_i, \text{Hom}(M_1, M_2)) = \text{Hom}_{kG}(k_i \otimes M_1, M_2)$. Since M_1, M_2 and hence also $k_i \otimes M_1$ are simple it follows that $k_i \otimes M_1 \cong M_2$.

5. Nilpotent groups

Here we prove Theorem D. In order to prove (ii) \Rightarrow (i) we invoke Theorem B. We consider the partition $\mathfrak{M} = \mathfrak{M}' \cup \mathfrak{M}''$ of the set of isomorphism classes of simple kG -modules where \mathfrak{M}' consists of the trivial module only and \mathfrak{M}'' consist of all non-trivial simple kG -modules. Then $\text{Ext}_{kG}^1(k, M'') = H^1(G, M'') = 0$ so that the partition satisfies (H). Since $C(k) = G$ we have $H^1(G/C(k), k) = 0$. Thus it follows from Theorem B that G is p -solvable and all its p -chief factors are isomorphic to k . Hence G is p -nilpotent (see [10], p. 428).

Since (iii) \Rightarrow (ii) is trivial, it remains to prove (i) \Rightarrow (iii). Thus let G be p -nilpotent and let M be a non-trivial simple kG -module. Then G has a normal p -complement N , so that G is the split extension of N and a p -Sylow subgroup P . We then consider the spectral sequence

$$H^r(P, H^s(N, M)) \Rightarrow H^{r+s}(G, M).$$

Since $|N|$ and $|M|$ are relatively prime the spectral sequence collapses and we obtain

$$H^r(G, M) = H^r(P, H^0(N, M)) = H^r(P, M^N).$$

Since M is simple we have $M^N = 0$ or $M^N = M$. In the latter case N would operate trivially on M , so that M would be a simple $k(G/N) = kP$ module, i.e. we would

have $M = k$. But this is a contradiction since we have assumed that M is non-trivial. Thus $M^N = 0$ and hence $H^r(G, M) = 0$, for $r \geq 0$.

We remark that it follows from Theorem D that a group G is p -nilpotent if and only if the principal block of kG consists of the trivial module only. This is also a well-known result in modular representation theory (see [1], p. 157).

Corollary 4. *A finite group G is p -nilpotent if and only if the principal block of kG consists of the trivial module k only.*

As a further application we also prove the following generalization of Corollary 4 (see [13], or [11], p. 545).

Corollary 5.² *Let G be p -nilpotent with $p \nmid |G|$. Then every block of kG contains at most one isomorphism class of simple kG -modules.*

Proof. Suppose one block contains two non-isomorphic simple kG -modules. Then we conclude from Proposition 2 that there are two simple modules M_1, M_2 with $M_1 \not\cong M_2$ and $\text{Ext}_{kG}^1(M_1, M_2) \neq 0$. Hence $H^1(G, \text{Hom}(M_1, M_2)) \neq 0$. By Corollary 4 it follows that $M = \text{Hom}(M_1, M_2)$ contains k as a composition factor. But then $M^G = \text{Hom}_{kG}(M_1, M_2) \neq 0$, by Lemma 3 below, so that $M_1 \cong M_2$.

Lemma 3. *Let G be a p -nilpotent group with $p \nmid |G|$, and let M be a kG -module containing k as composition factor. Then $M^G = H^0(G, M) \neq 0$.*

Proof. We proceed by induction on the length of the composition series of M . If M is simple, then $M = k$, so that $H^0(G, M) = k$. If M is not simple, let \bar{M} be a simple submodule and consider the extension $\bar{M} \rightarrow M \rightarrow M/\bar{M}$. Then we have an exact sequence

$$0 \rightarrow H^0(G, \bar{M}) \rightarrow H^0(G, M) \rightarrow H^0(G, M/\bar{M}) \rightarrow H^1(G, \bar{M}) \rightarrow \dots$$

If $\bar{M} \cong k$ then $H^0(G, \bar{M}) = k$ and $H^0(G, M) \neq 0$. If $\bar{M} \not\cong k$ then $H^1(G, \bar{M}) = 0$ by Theorem D. Since M/\bar{M} has k as composition factor and has a shorter composition series than M , we have $H^0(G, M/\bar{M}) \neq 0$ by induction. Hence $H^0(G, M) \neq 0$. This completes the proof of Lemma 3.

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² Of course Corollary 5 could also be obtained from Corollaries 3 and 4.

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